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Non-archimedean improper measures on homogeneous spaces

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INTRODUCTION

Let $\mathscr{K}$ be a complete non-archimedean valued field.

In the first part of this paper we will prove some natural isomorphisms in the category of non-archimedean Banach modules over $\mathscr{K}$.

In the second part of this paper we will give an application.

Let G be a locally compact zerodimensional group.

We suppose that G has a $\mathscr{K}$-valued invariant measure.

(For information about invariant measures on groups: see [1], 8.1–8.4.)

Let H be a closed subgroup of G and let G/H be the homogeneous space of all left cosets of H in G.

In [3] it is proved that G/H has a $\mathscr{K}$-valued quasi-invariant measure.

Let $M^\infty(G/H)$ be the space of all bounded measures on G/H.

We will define the space $M_\infty(G/H)$ of improper measures on G/H.

(For information about improper measures on groups: see [1], 8.B.)

$M^\infty(G/H)$ and $M_\infty(G/H)$ can be seen as $L(G)$-modules where $L(G)$ is the group algebra of G.

(For information about group algebra modules: see [5] and [6].)

Using the natural isomorphisms of the first part of the paper we obtain natural isomorphisms:

$$\mathrm{HOM}^L_{L(G)}(L(G), M^\infty(G/H)) \cong M^\infty(G/H)$$

and

$$\mathrm{HOM}^L_{L(G)}(L(G), M_\infty(G/H)) \cong M_\infty(G/H).$$

The essential part of $M^\infty(G/H)$ is isomorphic with $BUC(G/H)$ (see [2] and [4]), the space of all uniformly continuous functions on G/H.

In the same way the essential part of $M_\infty(G/H)$ is isomorphic with $C_\infty(G/H)$, the space of all continuous functions on G/H which vanish at infinity.

Therefore we also obtain natural isomorphisms:

$$\mathrm{HOM}^L_{L(G)}(L(G), BUC(G/H)) \cong M^\infty(G/H)$$

and

$$\mathrm{HOM}^L_{L(G)}(L(G), C_\infty(G/H)) \cong M_\infty(G/H).$$

1. NON-ARCHIMEDEAN BANACH MODULES

Let $\mathscr{K}$ be a *complete non-archimedean valued field.*

Let E be a *non-archimedean Banach space* over $\mathscr{K}$.

Let A be a *non-archimedean Banach algebra* over $\mathscr{K}$.

We say that E is a *left* (resp. *right*) *Banach A-module* if there exists a multiplication $(\lambda, x) \to \lambda x$ (resp. $(\lambda, x) \to x\lambda$) $(\lambda \in A, x \in E)$ such that $\|\lambda x\| \leq \|\lambda\| \, \|x\|$ (resp. $\|x\lambda\| \leq \|x\| \, \|\lambda\|$).

We say that E is a *Banach A-bimodule* if it is both a left and a right Banach A-module and if

$$(\lambda x)\mu = \lambda(x\mu) \quad (x \in E,\ \lambda \in A,\ \mu \in A).$$

In a natural way A itself is a Banach A-bimodule.

In what follows we will suppose that A has an *approximate identity.* By this we mean that there is a net $(\lambda_\alpha)_{\alpha \in I}$ in A with the following properties:

a) $\forall \alpha \in I \ \|\lambda_\alpha\| \leq 1$,
b) $\lim_{\alpha \in I} \lambda_\alpha \lambda = \lim_{\alpha \in I} \lambda \lambda_\alpha = \lambda \ (\lambda \in A)$.

If E is a left (resp. right) Banach A-module then we define $E_e := \mathrm{clo}\ (AE)$ (resp. $E_e := \mathrm{clo}\ (EA)$). E_e will be called the *essential part* of E.

It is clear that x is an element of E_e if and only if $\lim_{\alpha \in I} \lambda_\alpha x = x$ (resp. $\lim_{\alpha \in I} x\lambda_\alpha = x$).

We say that E is an *essential* left (resp. right) Banach A-module if $E = E_e$.

If E and F are two left (resp. right) Banach A-modules then $\mathrm{HOM}^L_A(E, F)$ (resp. $\mathrm{HOM}^R_A(E, F)$) will be the space of all *continuous linear maps* $T: E \to F$ for which $T(\lambda x) = \lambda T(x)$ (resp. $T(x\lambda) = T(x)\lambda$) $(\lambda \in A,\ x \in E)$.

We say that E and F are *isomorphic* (notation $E \cong F$) if there exists a surjective linear isometry in $\mathrm{HOM}^L_A(E, F)$ (resp. $\mathrm{HOM}^R_A(E, F)$).

A closed subspace F of a left (resp. right) Banach A-module E is a *submodule* of E if it is itself a Banach A-module.

Such a submodule will be called *weakly essential* if we have the following property:

> If x is an element of E such that $Ax \subset F$ (resp. $xA \subset F$) then x has to be an element of F.

(Notice that if $F = F_e$ then of course F is weakly essential!)

2. CONSTRUCTION OF BANACH MODULES

Let F be a left (resp. right) Banach A-module and let E be a Banach A-bimodule.

$\mathrm{HOM}_A^L(E, F)$ (resp. $\mathrm{HOM}_A^R(E, F)$) can be made into a left (resp. right) Banach A-module by setting:

$$(\lambda T)(x) := T(x\lambda) \text{ (resp. } (T\lambda)(x) := T(\lambda x))$$

$$(T \in \mathrm{HOM}_A^L(E, F) \text{ (resp. } T \in \mathrm{HOM}_A^R(E, F)),\ \lambda \in A,\ x \in E).$$

If E is a left (resp. right) Banach A-module then we have a natural isomorphism of Banach A-modules

$$[\mathrm{HOM}_A^L(A, E)]_e \cong E_e \text{ (resp. } [\mathrm{HOM}_A^R(A, E)]_e \cong E_e).$$

Let E be a left (resp. right) Banach A-module and let F be a right (resp. left) Banach A-module. In a natural way we can define a *tensor product* $E \hat{\otimes}_A^R F$ (resp. $E \hat{\otimes}_A^L F$) which always exists. (The definition and the construction of the tensor product run exactly the same as in [5]).

Let E be a left (resp. right) Banach A-module and let F be a Banach A-bimodule.

$E \hat{\otimes}_A^R F$ (resp. $E \hat{\otimes}_A^L F$) can be made into a left (resp. right) Banach A-module by setting:

$$\lambda(x \otimes_A y) := x \otimes_A \lambda y \text{ (resp. } (x \otimes_A y)\lambda := x \otimes_A y\lambda) \ (x \in E,\ y \in F,\ \lambda \in A).$$

If E is a left (resp. right) Banach A-module then we have a natural isomorphism of Banach A-modules.

$$E \hat{\otimes}_A^R A \cong E_e \text{ (resp. } E \hat{\otimes}_A^L A \cong E_e).$$

(We leave it to the reader to see what happens when F is a right (resp. left) Banach A-module and E is a Banach A-bimodule.)

Let D be a left (resp. right) Banach A-module.

Let E and F be two Banach A-bimodules.

We have a natural isomorphism of left (resp. right) Banach A-modules

$$\mathrm{HOM}_A^L(E \hat{\otimes}_A^L F, D) \cong \mathrm{HOM}_A^L(F, \mathrm{HOM}_A^L(E, D))$$

$$\text{(resp. } \mathrm{HOM}_A^R(E \hat{\otimes}_A^R F, D) \cong \mathrm{HOM}_A^R(F, \mathrm{HOM}_A^R(E, D))).$$

Let E be a right (resp. left) Banach A-module.

The *dual* Banach space E^* of E can be given a left (resp. right) Banach A-module structure simply by looking at it as $\mathrm{HOM}_A^L(E, \mathscr{K})$ (resp. $\mathrm{HOM}_A^R(E, \mathscr{K})$)

where we give both E and $\mathscr{K}$ the trivial left (resp. right) Banach A-module structure.

After all these (somewhat categorical) definitions and isomorphisms we are able to formulate a first proposition.

3. A NATURAL ISOMORPHISM

3. PROPOSITION

Let E be an essential right (resp. left) Banach A-module.

Let F be a weakly essential submodule of the left (resp. right) Banach A-module E^*.

We have a natural isomorphism of left (resp. right) Banach A-modules.

$$\mathrm{HOM}_A^L(A,F) \cong F \text{ (resp. } \mathrm{HOM}_A^R(A,F) \cong F).$$

PROOF:

(We give the proof for an essential right Banach A-module E). We have a natural isomorphism:

$$E^* \cong \mathrm{HOM}_A^L(E,\mathscr{K}) \cong \mathrm{HOM}_A^L(E\hat{\otimes}_A^L A,\mathscr{K})$$

$$\cong \mathrm{HOM}_A^L(A,\mathrm{HOM}_A^L(E,\mathscr{K})) \cong \mathrm{HOM}_A^L(A,E^*).$$

This natural isomorphism is such that every x in E^* corresponds to the element $\lambda \to \lambda x$ of $\mathrm{HOM}_A^L(A,E^*)$.

F being a weakly essential submodule of E^*, we see that this natural isomorphism also induces a natural isomorphism $F \cong \mathrm{HOM}_A^L(A,F)$.

4. AN EXAMPLE

4.1. DEFINITIONS

Let $\mathscr{K}$ be a complete non-archimedean valued field.

Let X be a *locally compact zerodimensional space* (see [1], p. 39).

Let $\mathscr{B}_c(X)$ be the ring of all *open compact* subsets of X.

Let $BC(X)$ be the space of all *bounded continuous* $\mathscr{K}$-valued functions on X.

Let $BUC(X)$ be the space of all *bounded $\mathscr{U}$-uniformly continuous* $\mathscr{K}$-valued functions X where $\mathscr{U}$ is a *non-archimedean uniformity* on X compatible with the topology on X (see [1], p. 34).

Let $C_\infty(X)$ be the space of all continuous $\mathscr{K}$-valued functions on X which *vanish at infinity*.

With the supnorm $BC(X)$, $BUC(X)$ and $C_\infty(X)$ are non-archimedean Banach spaces over $\mathscr{K}$.

Let $M^\infty(X)$ be the space of all *bounded additive* $\mathscr{K}$-valued functions on $\mathscr{B}_c(X)$. With the supnorm $M^\infty(X)$ is a non-archimedean Banach space over $\mathscr{K}$. The elements of $M^\infty(X)$ are the so-called *bounded measures* on X. $M^\infty(X)$ is the dual space of $C_\infty(X)$; the duality is given by

$$(f,\lambda) \to \int_X f\,\mathrm{d}\lambda = \int_X f(x)\,\mathrm{d}\lambda(x) \ (f \in C_\infty(X),\ \lambda \in M^\infty(X)).$$

(see [1], 7c).

4.2. IMPROPER MEASURES MODULO H ON G

Let G be a *locally compact zerodimensional group.*

Let λ be an element of $M^\infty(G)$. If s is an element of G we can define an element λ_s of $M^\infty(G)$ by setting:

$$\int_G f\,\mathrm{d}\lambda_s := \int_G L_s f\,\mathrm{d}\lambda \quad (f\in C_\infty(G))$$

where $L_s f(t) := f(st)$ $(t\in G)$.

If f is an element of $C_\infty(G)$ we can define an element $f*\lambda$ of $M^\infty(G)$ by setting:

$$\int_G g\mathrm{d}\, f*\lambda := \int_G f(s)[\int_G g\mathrm{d}\lambda_s]\mathrm{d}s \quad (g\in C_\infty(G)).$$

(Here $\mathrm{d}s$ stands for $\mathrm{d}m(s)$ where m is an invariant measure on G (i.e. $m_s=m$ for every s in G)).

In [4] it is proved that $f*\lambda=Fm$ where

$$F(s)=\int_G \Delta(s)^{-1}f(ts^{-1})\,\mathrm{d}\lambda(t).$$

(Here Δ stands for the modular function on G).

It is also proved there that F is an element of $BRUC(G)$, the space of all *bounded right uniformly continuous* $\mathscr{K}$-valued functions on G.

(The "right group uniformity" on G is the one which is generated by the partitions consisting of the right cosets of the open compact subgroups of G.)

Let H be a *closed subgroup* of G and let $\pi: G\to G/H$ be the natural quotient map onto the *space of all left cosets of H in G.*

Let $C^H_{\infty,R}(G)$ be the space of all functions f of $BRUC(G)$ such that for every $\varepsilon>0$ there exists a compact subset K of G such that for every $s\notin KH$ we have $|f(s)|\le\varepsilon$.

Let $M^H_{\infty,R}(G)$ be the space of all measures λ of $M^\infty(G)$ such that for every f in $C_\infty(G)$ the element F of $BRUC(G)$ for which $f*\lambda=Fm$ actually belongs to $C^H_{\infty,R}(G)$. The elements of $M^H_{\infty,R}(G)$ are called *improper measures modulo H* on G. In the next proposition we give a characterisation of the elements of $M^H_{\infty,R}(G)$.

4.3. PROPOSITION

Let λ be an element of $M^\infty(G)$. λ belongs to $M^H_{\infty,R}(G)$ if and only if the following holds: for every open compact subgroup K of G and every $\varepsilon>0$ there exist finitely many elements $(s_k)_{1\le k\le n}$ of G such that for every s in G for which $\pi(Ks)$ is not equal to one of the $\pi(Ks_k)$ $(1\le k\le n)$ we have $|\lambda(Ks)|\le\varepsilon$.

PROOF:

"$\Rightarrow$" An easy computation gives $\xi(K)*\lambda=Fm$ with $F(s)=\Delta(s)^{-1}\lambda(Ks)$ $(s\in G)$. (As usual we have used the notation $\xi(A)$ to denote the characteristic function of a subset A of G). If $\varepsilon>0$ is chosen there exists a compact subset K_ε of G such that for every $s\notin K_\varepsilon H$ we have $|F(s)|\le\varepsilon$ and therefore also $|\lambda(Ks)|\le\varepsilon$. It is clear that there exist finitely many elements $(s_k)_{1\le k\le n}$ of G such that $\pi(K_\varepsilon)$ is covered by the $(\pi(Ks_k))_{1\le k\le n}$. If $\pi(Ks)$ is not equal to one of those $\pi(Ks_k)$ $(1\le k\le n)$, then $s\notin K_\varepsilon H$ and therefore $|\lambda(Ks)|\le\varepsilon$.

"$\Leftarrow$" It suffices to prove that for every open compact subgroup K of G and every t in G the element F of $BRUC(G)$ for which $\xi(Kt)*\lambda=Fm$ belongs to $C^R_{\infty,H}(G)$. If $\varepsilon>0$ then there exist finitely many elements $(s_k)_{1\le k\le n}$ of G such that for every s in G for which $\pi(Ks)$ is not equal to one of the $\pi(Ks_k)$ we have $|\lambda(Ks)|\le\varepsilon$. Now $F(s)=\Delta(t)\Delta(s)^{-1}\lambda(Kt^{-1}s)$. Consider $\bigcup_{k=1}^n \pi(tKs_k)$. This is a compact subset of G/H. There exists a compact subset K_ε of G such that for every $s\notin K_\varepsilon H$ we have $\pi(s)\notin\bigcup_{k=1}^n \pi(tKs_k)$, which means that $\pi(Kt^{-1}s)$ is not equal to one of the $\pi(Ks_k)$ and we see that $|F(s)|=|\lambda(Kt^{-1}s)|\le\varepsilon$.

4.4. IMPROPER MEASURES ON G/H

Let G be a locally compact zerodimensional group.

Let H be a closed subgroup of G. Let $\pi:G\to G/H$ be the natural quotient map from G onto the space of all left cosets of H in G. With the *quotient topology* G/H is also a locally compact zerodimensional space.

Let λ be an element of $M^\infty(G/H)$. If s is an element of G we can define an element λ_s of $M^\infty(G/H)$ by setting:

$$\int_{G/H} f\,\mathrm{d}\lambda_s := \int_{G/H} L_s f\,\mathrm{d}\lambda \quad (f\in C_\infty(G/H))$$

where $L_s f(\pi(t)) := f(\pi(st))$ $(\pi(t)\in G/H)$.

If f is an element of $C_\infty(G)$ we can define an element $f*\lambda$ of $M^\infty(G/H)$ by setting:

$$\int_{G/H} g\mathrm{d}\,f*\lambda := \int_G f(s)[\int_{G/H} g\mathrm{d}\lambda_s]\mathrm{d}s \quad (g\in C_\infty(G/H)).$$

In [4] it is proved that $f*\lambda=F\mu$ where μ is a quasi-invariant measure on G/H (see [3]) and where F is an element of $BUC(G/H)$, the space of all *bounded uniformly continuous* $\mathscr{K}$-valued functions on G/H.

(The "homogeneous" uniformity on G/H is the one which is generated by the partitions consisting of the images by π of the right cosets of the open compact subgroups of G.)

Let $M_\infty(G/H)$ be the space of all measures λ of $M^\infty(G/H)$ such that for every f in $C_\infty(G)$ the element F of $BUC(G/H)$ for which $f*\lambda=F\mu$ actually belongs to $C_\infty(G/H)$.

The elements of $M_\infty(G/H)$ are called *improper measures* on G/H. Of course there is a link between the elements of $M^H_{\infty,R}(G)$ and the elements of $M_\infty(G/H)$: if λ is an element of $M^\infty(G/H)$ we can define an element $\lambda^\#$ of $M^\infty(G)$ by setting:

$$\int_G f\,\mathrm{d}\lambda^\# := \int_{G/H} f^b\,\mathrm{d}\lambda \quad (f\in C_\infty(G))$$

where

$$f^b(\pi(s)) = \int_H f(st)\,\mathrm{d}t;$$

now λ is an element of $M_\infty(G/H)$ if and only if $\lambda^\#$ is an element of $M^H_{\infty,R}(G)$. (For the details we refer to [2].)

The elements of $M_\infty(G/H)$ can also be characterised as in the following proposition:

4.5. PROPOSITION

Let λ be an element of $M^{\infty}(G/H)$. λ belongs to $M_{\infty}(G/H)$ if and only if the following holds: for every open compact subgroup K of G and every $\varepsilon < 0$ there exist finitely many elements $(s_k)_{1 \le k \le n}$ of G such that for every s in G for which $\pi(Ks)$ is not equal to one of the $\pi(Ks_k)$ $(1 \le k \le n)$ we have $|\lambda(\pi(Ks))| \le \varepsilon$.

PROOF:

See [2] for the details.

4.6. THE GROUP ALGEBRA OF G

Let f be an element of $C_{\infty}(G)$ and let s be an element of G. We can define an element $\mathscr{L}_s f$ (resp. $\mathscr{R}_s f$) of $C_{\infty}(G)$ by setting:

$$\mathscr{L}_s f(t) = f(s^{-1}t) \text{ (resp. } \mathscr{R}_s f(t) = \Delta(s)^{-1} f(ts^{-1})) \ (t \in G).$$

If f and g are two elements of $C_{\infty}(G)$ we see that

$$\int_G f(s) \mathscr{L}_s g \mathrm{d}s = \int_G g(s) \mathscr{R}_s f \, \mathrm{d}s.$$

(Here we use vector valued integration; see for instance [1] or [2]). This element of $C_{\infty}(G)$ will be denoted by $f*g$.

In what follows we will write $L(G)$ instead of $C_{\infty}(G)$.

With the supremumnorm $(L(G), \| \, \|, *)$ becomes a non-archimedean Banach algebra over $\mathscr{K}$. $L(G)$ will be called the *group algebra* of G. Let $(K_{\alpha})_{\alpha \in I}$ be a fundamental system of neighborhoods of the unit element of G consisting of open compact q-free subgroups (where q is the characteristic of the residue class field of $\mathscr{K}$). (An open compact subgroup K_1 of G is *q-free* if for every open compact subgroup K_2 of G with $K_2 \subset K_1$ the index $[K_1 : K_2]$ is not divisible by q).

If we define

$$\begin{aligned} u_{\alpha}(s) &= 0 & (s \notin K_{\alpha}), \\ u_{\alpha}(s) &= 1/m(K_{\alpha}) & (s \in K_{\alpha}), \end{aligned}$$

then $(u_{\alpha})_{\alpha \in I}$ is an approximate identity of $L(G)$ (see [6]).

4.7. NATURAL ISOMORPHISMS

$C_{\infty}(G/H)$ is an essential right Banach $L(G)$-module if we define:

$$g*f := \int_G f(s) L_s g \, \mathrm{d}s \ (f \in L(G), g \in C_{\infty}(G/H)).$$

$M^{\infty}(G/H)$ is a left Banach $L(G)$-module with $f*\lambda$ defined as in 4.4.

It is clear that this left Banach $L(G)$-module structure on $M^{\infty}(G/H)$ is exactly the same as the left Banach $L(G)$-module structure on $M^{\infty}(G/H)$ when we look at it as the dual $C_{\infty}(G/H)^*$ of $C_{\infty}(G/H)$.

Therefore (using 3) we may conclude that in a natural way

$$\mathrm{HOM}^{L}_{L(G)}(L(G), M^{\infty}(G/H)) \cong M^{\infty}(G/H).$$

But there is more! It is not difficult to see that $M_{\infty}(G/H)$ is a weakly essential submodule of $M^{\infty}(G/H)$.

Therefore (using 3) we may conclude that in a natural way

$$\mathrm{HOM}^L_{L(G)}(L(G), M_\infty(G/H)) \cong M_\infty(G/H).$$

In [4] it is shown that there exists a quasi-invariant measure μ on G/H such that $g \to g\mu$ is an isometry from $BUC(G/H)$ into $M^\infty(G/H)$.

It is not difficult to see (using the results in [4]) that the essential part of $M^\infty(G/H)$ corresponds to the image of this isometry.

The corresponding Banach $L(G)$-module structure on $BUC(G/H)$ is given by:

$$f * g := \int_G f(s) \mathscr{L}_s g \, ds \quad (f \in L(G), g \in BUC(G/H))$$

where

$$\mathscr{L}_s g(\pi(t)) := g(\pi(s^{-1}t)) \frac{\varrho(s^{-1}t)}{\varrho(t)} \quad (\pi(t) \in G/H)$$

and where ϱ is the invertible element of $BRUC(G)$ such that $\mu^\# = \varrho m$. (See [3] for more details).

It is clear that the natural isomorphism of 3 could also be written as $\mathrm{HOM}^L_A(A, F_e) \cong F$ (resp. $\mathrm{HOM}^R_A(A, F_e) \cong F$).

Therefore we may conclude that in a natural way

$$\mathrm{HOM}^L_{L(G)}(L(G), BUC(G/H)) \cong M^\infty(G/H).$$

It is not difficult to see that $g \to g\mu$ is an isometry from $C_\infty(G/H)$ into $M_\infty(G/H)$ and that the essential part of $M_\infty(G/H)$ corresponds with the image of this isometry.

Therefore we may conclude that in a natural way

$$\mathrm{HOM}^L_{L(G)}(L(G), C_\infty(G/H)) \cong M_\infty(G/H).$$

If $H = \{e\}$ we obtain

$$\mathrm{HOM}^L_{L(G)}(L(G), M^\infty(G)) \cong M^\infty(G),$$

$$\mathrm{HOM}^L_{L(G)}(L(G), M^\infty_R(G)) \cong M^\infty_R(G),$$

$$\mathrm{HOM}^L_{L(G)}(L(G), BRUC(G)) \cong M^\infty(G),$$

$$\mathrm{HOM}^L_{L(G)}(L(G), L(G)) \cong M^R_\infty(G).$$

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